Hybrid quantum-classical and quantum-inspired classical algorithms for solving banded circulant linear systems

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Problem Statement

Solve $Cx = \boldsymbol{b}$ where

$$C = \begin{pmatrix} c_0 & c_{N-1} & \cdots & c_{N-K} & & c_K & \cdots & c_1 \\ c_1 & c_0 & \ddots & & \ddots & & \ddots & c_K \\ \vdots & \ddots & \ddots & \ddots & & \ddots & & \\ c_K & & \ddots & \ddots & \ddots & & \ddots & \\ & \ddots & & \ddots & \ddots & \ddots & & c_{N-K} \\ c_{N-K} & & \ddots & & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & & \ddots & & \ddots & c_0 & c_{N-1} \\ c_{N-1} & \cdots & c_{N-K} & & c_K & \cdots & c_1 & c_0 \end{pmatrix}. \text{ Let } Q = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}$$

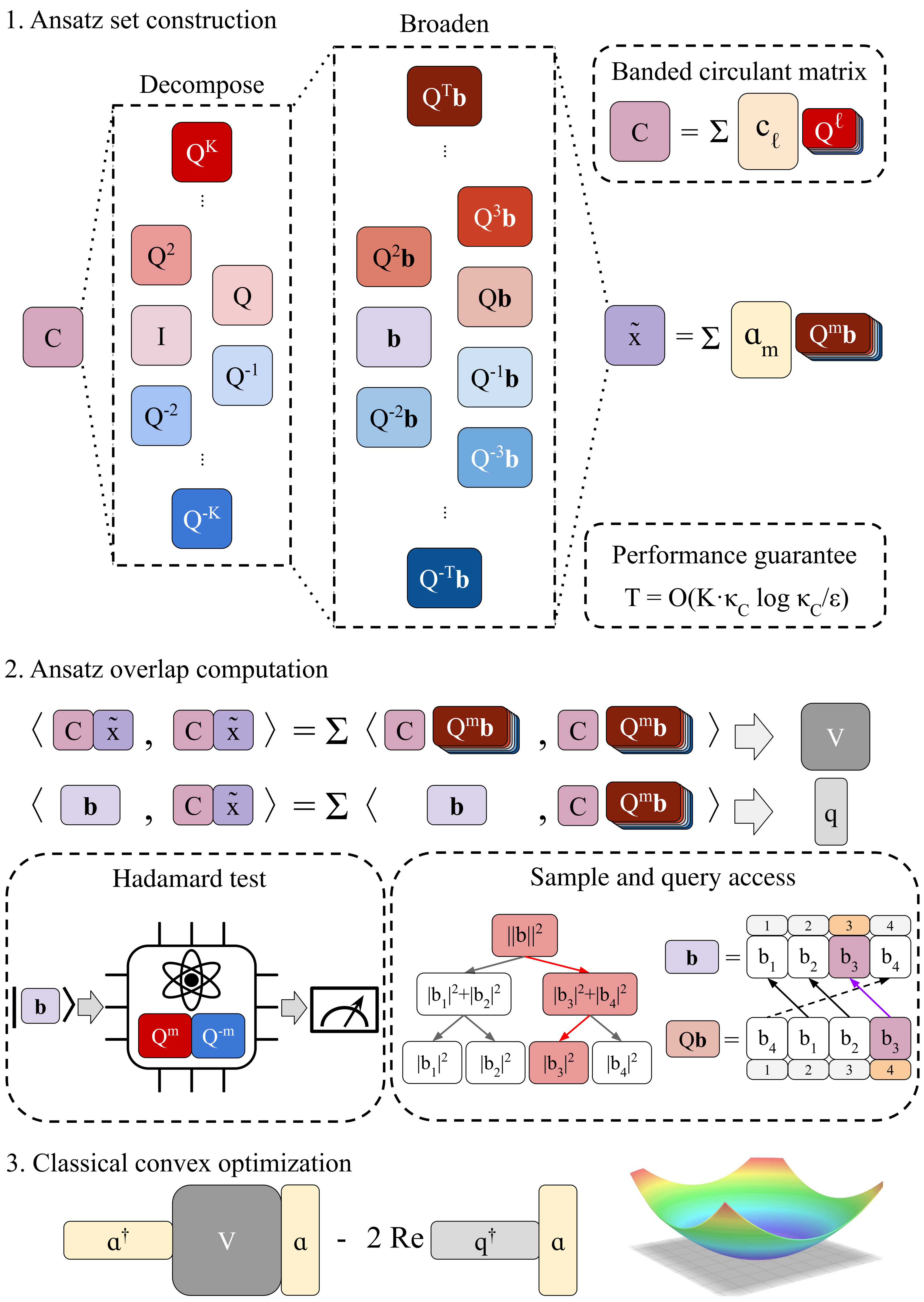
We can partition $C = \sum_{\ell=-K}^{K} c_\ell Q^\ell$.

Proposition 1. *Let $0 < \nu \leq 1$. Given a K-banded circulant matrix $C \in \mathcal{M}_N(\mathbb{C})$ where $\kappa_C = \|C\|\|C^{-1}\|$ and a normalized vector $\boldsymbol{b} \in \mathbb{C}^N$, there exists $T \in \mathcal{O}(K \cdot \kappa_C \log \frac{\kappa_C}{\nu})$ such that we can find an optimal set of parameters $\alpha = \{\alpha_m \in \mathbb{C}, \forall m \in [-T..T]\}$ and that the estimator $\tilde{x}(\alpha) = \sum_{m=-T}^{T} \alpha_m Q^m \boldsymbol{b}$ satisfies*

$$\min_{\alpha \in \mathbb{C}^{2T+1}} \|C\tilde{x}(\alpha) - \mathbf{b}\|^2 \leq \min_{x \in \mathbb{C}^N} \|Cx - \mathbf{b}\|^2 + \nu.$$

Given this proposition, we can construct a classical constructive algorithm that runs in $\mathcal{O}\left(K^{4.5}\kappa_C^3 \log^3 \frac{K^{0.5}\kappa_C}{\epsilon}\right)$ runtime based on directly generating Chebeshev coefficients shown in Childs et al. (2017) while truncating at a higher degree and constructing all relevant coefficients with an iterative algorithm. Given the similarities of the problem with Huang et al. (2021)'s near term quantum algorithm, it is interesting to see how the method performs in obtaining relevant coefficients with such a method.

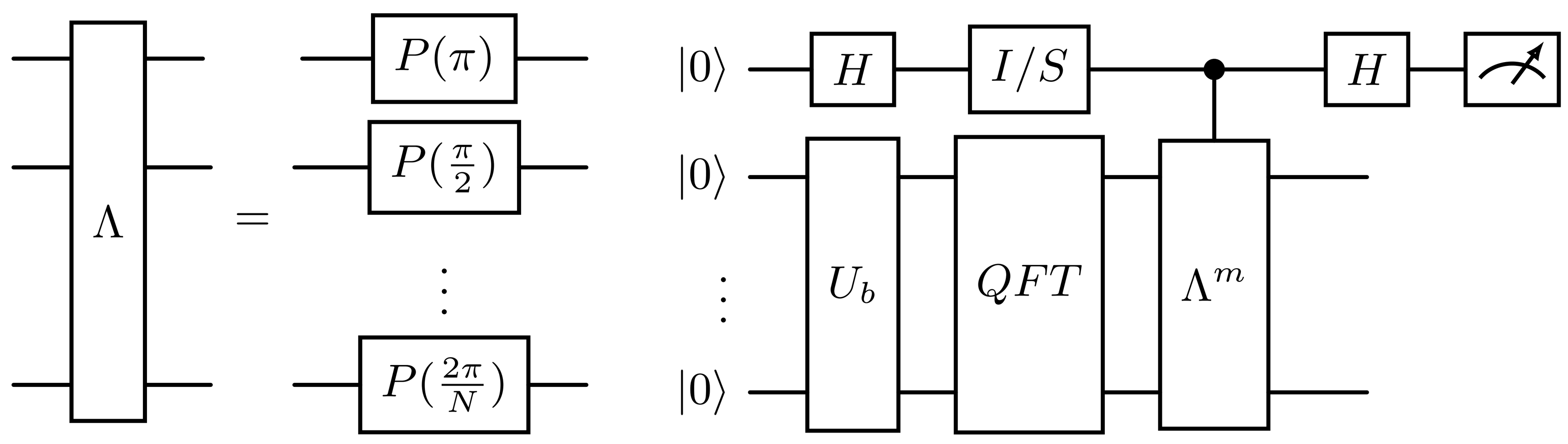
Main Method



Applying Huang et al. (2021)'s near-term algorithm for linear systems, we note Ansatz tree strategy collapses into two Krylov subspaces of $\mathcal{K}(Q, \boldsymbol{b})$ and $\mathcal{K}(Q^{-1}, \boldsymbol{b})$.

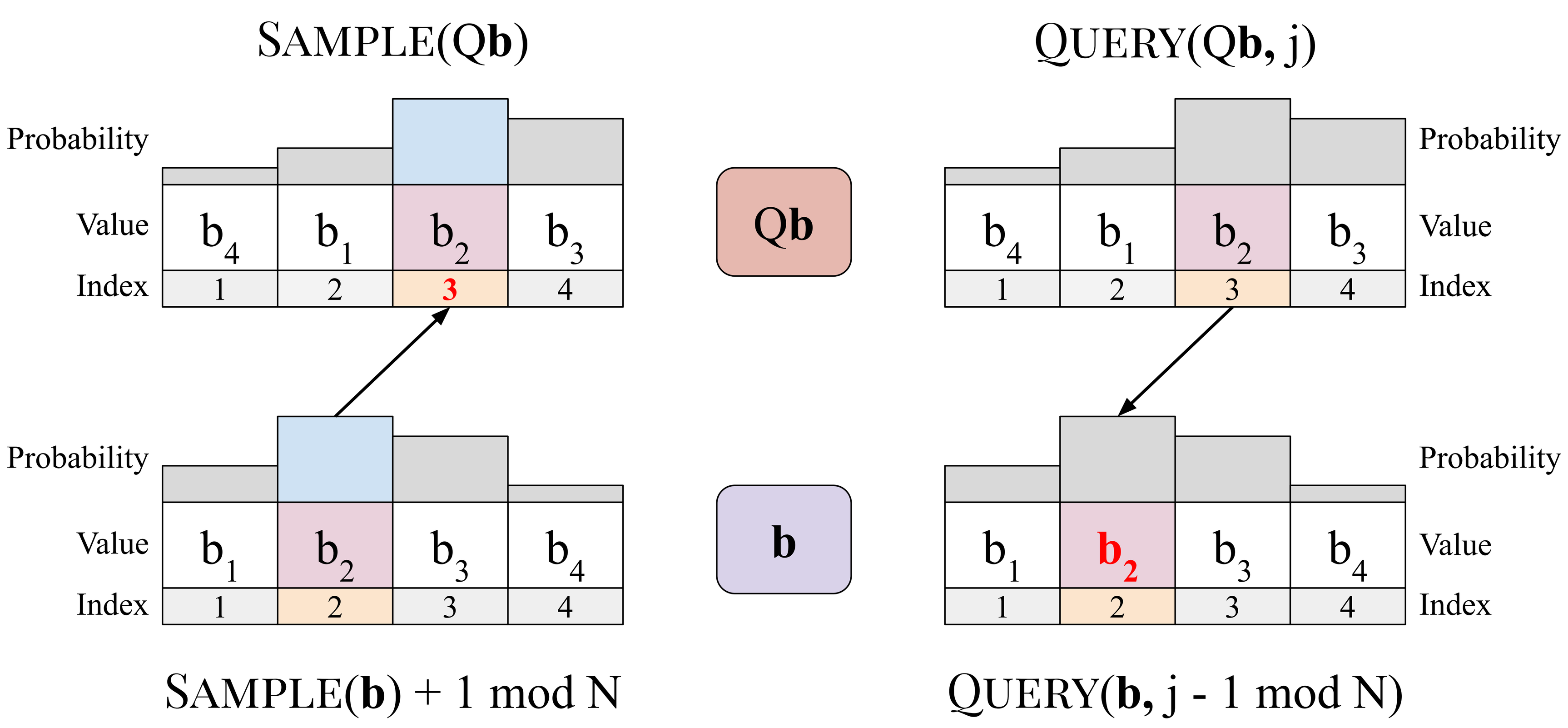
Implementation

Quantum circuits



The cyclic permutation matrix Q^m can be diagonalized by the QFT circuits F such that $Q = F^{-1}\Lambda^m F$. The total number of Hadamard tests required is $\mathcal{O}\left(\frac{K^2 B^4 \kappa_C^4}{\epsilon^2} \log^2 \frac{\kappa_C}{\epsilon}\right)$, where B is the ℓ_1 norm of the circulant coefficients c_i.

Classical sample and query access



SQ access is used for the inner product estimation algorithm by Tang (2019).

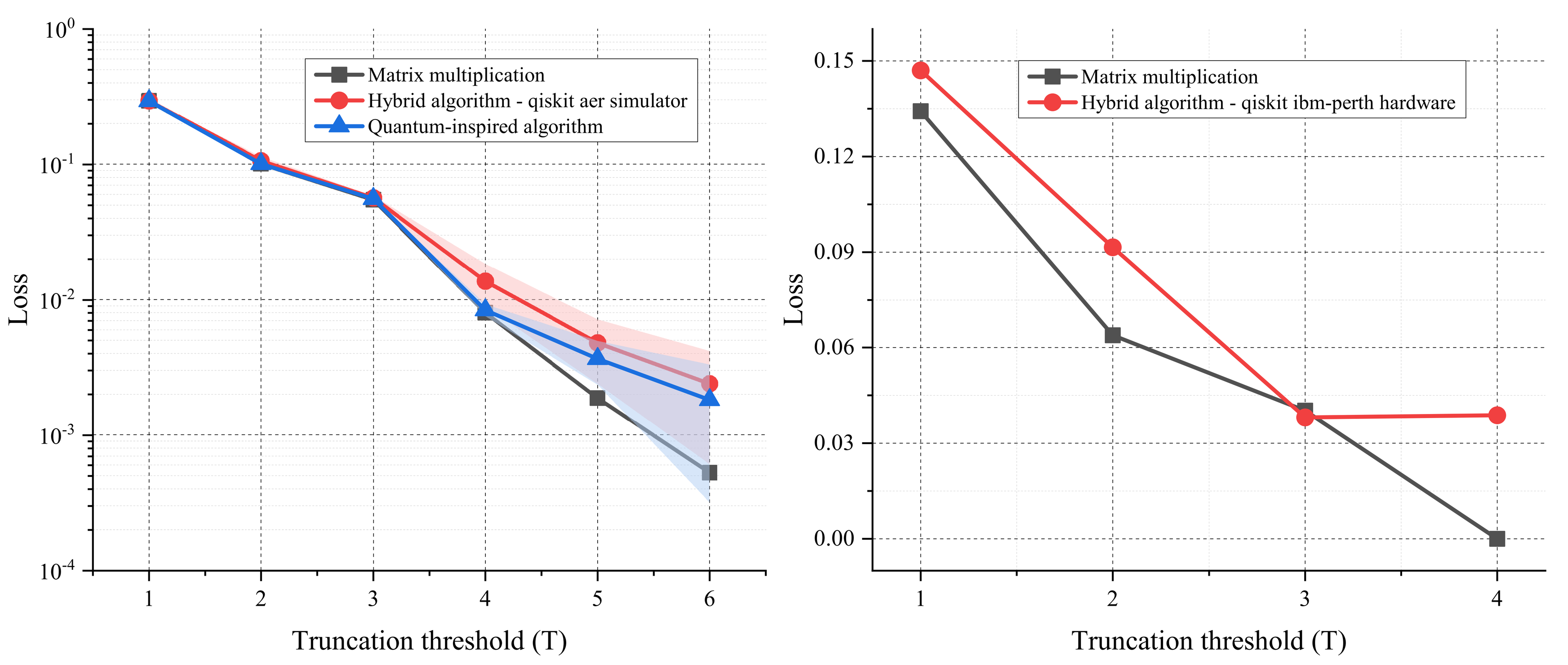
Hardness

Definition 2 (QuantumCirculantOptim). *We are given a K-banded circulant matrix $C \in \mathcal{M}_N(\mathbb{C})$, where $K \in \mathcal{O}(\operatorname{poly}\log N)$, as well a quantum circuit with oracle access with depth $\mathcal{O}(\operatorname{poly}\log N)$ that can generate given $\boldsymbol{b} \in \mathbb{C}^N$ as a $\log N$-qubit quantum state $|b\rangle$. Quantum circuits that generate the Ansatz set of $\log N$-qubit quantum states $\{|u_m\rangle\}_{m=1}^M$, and error $\zeta > 0$ are also provided. The problem is to find $\{\hat{\alpha}_m\}_{m=1}^M, \alpha_m \in \mathbb{C}$ such that*

$$\left\|C\sum_{m=1}^{M} \hat{\alpha}_m |u_m\rangle - |b\rangle\right\|^2 \leq \min_{\alpha \in \mathbb{C}^M} \left\|C\sum_{m=1}^{M} \alpha_m |u_m\rangle - |b\rangle\right\|^2 + \zeta.$$

Proposition 3. *If there exists a relativizing classical algorithm that solves QuantumCirculantOptim in $\operatorname{poly}(\kappa_C, \log N)$ time, then `PromiseBQP` = `PromiseBPP`.*

Experimental results



References

Childs A. M., Kothari R., Somma R. D., 2017, SIAM Journal on Computing, 46
Huang H.-Y., Bharti K., Rebentrost P., 2021, New Journal of Physics, 23
Tang E., 2019, in ACM Symposium on Theory of Computing 2019.

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